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SPHERICAL-TYPE HYPERSURFACES IN A RIEMANNIAN MANIFOLD

BY

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ABSTRACT

We extend some rigidity results of Aleksandrov and Ros on compact hypersurfaces in $\mathbf{R}^n$ to more general ambient spaces with the aid of the notion of almost conformal vector fields. These latter, at least locally, always exist and allow us to find interesting integral formulas fitting our purposes.

1. Introduction

Let M be a compact, oriented, hypersurface immersed in $\mathbf{R}^n$ and let $\mathcal{H}$ and $\mathcal{S}$ be its mean-curvature function and scalar curvature respectively. A classical global problem concerning these two geometrical quantities can be formulated as follows: *Assume that either $\mathcal{H}$ or $\mathcal{S}$ is constant. Is it true, possibly under some additional assumptions, that M is a sphere?* A celebrated result of Aleksandrov [A1] states that, assuming the immersion to be an embedding, the constancy of $\mathcal{H}$ implies M to be spherical. On the other hand Hsiang, Teng and Yu [H–T–Y] have proven, exhibiting a special class of hypersurfaces of $\mathbf{R}^{2n}$, that this latter condition alone is in general not sufficient to achieve the conclusion. The case of an immersed surface in $\mathbf{R}^3$ was firstly considered by Hopf [H], who positively answered the question assuming M to be topologically a 2-sphere, but Wente [W] has shown that the conclusion is false for surfaces of higher genus.

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For the scalar curvature the problem has been considered by a number of authors, notably by Cheng and Yau [C-Y], and recently Ros [RA] has obtained the following remarkable characterization: *The sphere is the only compact hypersurface with constant scalar curvature embedded in Euclidean space.* Ros' proof rests on a modification of an argument, due to Reilly, for proving Aleksandrov's theorem [RR1]. In particular this technique is based on some integral formulas for submanifolds of $\mathbf{R}^n$.

In this paper we give a generalization of Aleksandrov and Ros' results when the ambient space is an appropriate Riemannian manifold (N, h), and consider related problems. Towards this end let U be an open set in N. In [R-S], Rigoli and Salavessa introduced the following:

DEFINITION. A vector field X on U is said to be almost conformal if there exist smooth functions $\alpha, \beta : U \to \mathbf{R}$ such that the Lie derivative $L_X h$, of the metric h with respect to X, satisfies

$$2\alpha h \leq L_X h \leq 2\beta h. \tag{1.1}$$

X is said to be positive in case $\alpha > 0$.

For instance, on U a conformal vector field X is almost conformal with $\alpha = \beta$, and any homothetic vector field for which $L_X h$ is positive definite is positive almost conformal. On $\mathbf{R}^n$ with its canonical metric $\langle \ , \ \rangle$, the position vector field X is positive homothetic satisfying $L_X \langle \ , \ \rangle = 2 \langle \ , \ \rangle$.

A useful procedure to construct almost conformal vector fields is obtained by observing that given any real function $\phi : U \to \mathbf{R}$, the gradient vector field X satisfies (1.1) if and only if

$$\alpha h \leq \operatorname{Hess}(\phi) \leq \beta h.$$

As a consequence, indicating by ρ the distance function from a point p in the complete manifold (N, h), the vector field $X = \rho \partial / \partial \rho$ is positive almost conformal on any regular ball $U = B_R(p)$ centred at p. This follows from [H-K-W] (see also Th. 5.2 of [Hi]) as reported in [R-S]. Indeed if $\kappa = \max\{0, \sup_{B_R(p)} K\}$, K the sectional curvatures of N, and $K \geq \omega$, $\omega \leq 0$ on $B_R(p)$, we have

$$a_\kappa(\rho) h \leq \operatorname{Hess}(\rho^2/2) \leq a_\omega(\rho) h,$$

with $a_\kappa(t) = t\sqrt{\kappa} \cot(\sqrt{\kappa} t)$ for $0 \leq t < \pi/\sqrt{\kappa}$, $a_\omega(t) = t\sqrt{-\omega} \coth(\sqrt{-\omega} t)$ for $0 \leq t < +\infty$. By the Cartan-Hadamard theorem this is particularly significant if N

is simply connected with non-positive sectional curvatures, since in this case any geodesic ball is regular.

The class of (positive) almost conformal vector fields is therefore large enough to justify their use and in [R–S] they revealed themselves as an appropriate tool in determining, for instance, necessary and sufficient conditions for a given immersion $f:(M,g)\to(N,h)$ to be an isometry. In the present paper they will be used to derive integral formulas fitting our purposes.

The following results are typical examples of those that we obtain with this technique. In the sequel the manifolds B and its boundary M will always be tacitly assumed to be compact, connected, oriented, being the orientation of M, the usual induced one. Furthermore, given the almost conformal vector field X on U we let for $f:B\to U\subset N$, $\tilde f = f_{/M}:M\to U$,

$$a=\inf_{f(B)}\alpha,\quad b=\inf_{f(B)}\beta,\quad \tilde a=\inf_{\tilde f(M)}\alpha,\quad \tilde b=\inf_{\tilde f(M)}\beta.$$

Proposition A. *Let $f:(B,g)\to U\subset(N,h)$ be an isometric immersion into an open set supporting a positive almost conformal vector field X and let $n=\dim B=\dim N$. Assume that $\tilde f:M\to N$ has constant mean curvature. Then, letting $\tilde H$ denote the mean-curvature vector of $\tilde f$, the following isoperimetric inequality holds:*

$$n\frac{a}{\tilde b}\|\tilde H\|\le\frac{V(M)}{V(B)}\le n\frac{b}{\tilde a}\|\tilde H\|, \tag{1.2}$$

where $V(M)$ and $V(B)$ are the volumes of M and B relative to the metrics g and $\tilde g=g_{/M}$, respectively.

Observe that in case X is a positive homothetic vector field, then the above inequalities reduce to $V(M)=n\|\tilde H\|V(B)$, a well-known result for instance in $(N,h)=(\mathbf{R}^n,\langle\ ,\ \rangle)$.

We denote by dV_g and $dV_{\tilde g}$ the volume elements of (B,g) and $(M,\tilde g)$, respectively. Let $E(t)=\frac{1}{2}\int_B|\nabla t|^2\,dV_g$ be the energy of $t:B\to\mathbf{R}$.

Theorem B. *Let $f:(B,g)\to U\subset(N,h)$ be an isometric immersion into an open set supporting a positive almost conformal vector field X and let $n=\dim B=\dim N$. Assume that N is Einstein with scalar curvature $\mathcal{R}$ and that $\tilde f:M\to N$ has constant mean curvature. Then the solution t of the Poisson problem*

$$\Delta t=(a/b)^{1/2}\quad\text{on } B,\qquad t_{/M}=0, \tag{$\mathcal{P}$}$$

satisfies

$$\mathcal{R}E(t) \leq a(n-1)\frac{b-a}{2b^2}V(B). \tag{1.3}$$

If equality holds, then B is isometric to a metric ball in $\mathbf{R}^n$.

Observe that as a consequence of (1.3), if X is a positive homothetic vector field on U then $\mathcal{R} \leq 0$ because otherwise t would be constant and ($\mathcal{P}$) would imply $a = 0$, contradicting the fact that X is positive. The theorem generalizes Aleksandrov's result. Indeed it reduces, in case $(N,h) = (\mathbf{R}^n, \langle\ ,\ \rangle)$, to a stronger version due to Aleksandrov himself [A2]. This follows by choosing U and the homothetic vector field X to be $\mathbf{R}^n$ together with the position vector, since in this case equality in (1.3) is automatically achieved.

We state next results in case X is a conformal vector field. In section 2 we obtain similar results under different assumptions, in case X is almost conformal. Recall that given a compact domain B with boundary M in an oriented Riemannian manifold N, an *elliptic point* $m \in M$ is a point at which $g(\nabla di_m(Y,Y), (\nu_M)_m) < 0$ for each $Y \in T_mM$, $Y \neq 0$, Here, i is the inclusion $M \subset B$, ∇di_m is its second fundamental form and $(\nu_M)_m$ is the outward unit normal of M at m.

PROPOSITION C. *Let* $f:(B,g) \to U \subset (N,h)$ *be an isometric immersion into an open set supporting a positive conformal vector field X and let* $n = \dim B = \dim N \geq 3$. *Assume that N is Einstein with scalar curvature* $\mathcal{R}$, *and that* $\tilde{f}: M \to N$ *has constant scalar curvature* $\mathcal{S}$ *satisfying*

$$\mathcal{S} - \frac{n-2}{n}\mathcal{R} > 0, \tag{1.4}$$

and that there exists at least one elliptic point on M. Then the following isoperimetric inequality holds:

$$\frac{V(M)}{V(B)} \leq \frac{b}{\tilde{a}}\frac{n}{(n-1)^{1/2}(n-2)^{1/2}}\left(\mathcal{S} - \frac{n-2}{n}\mathcal{R}\right)^{1/2}. \tag{1.5}$$

If equality holds in (1.5) *then* $\tilde{f}: M \to N$ *is totally umbilical.*

Observe that (1.5) is comparable to the second half of (1.2). Indeed, in the assumptions of Proposition C we have

$$\left(\mathcal{S} - \frac{n-2}{n}\mathcal{R}\right)^{1/2} \leq (n-1)^{1/2}(n-2)^{1/2}\inf_M \|\tilde{H}\|$$

(see Eq. (2.14) of section 2). A first consequence of Proposition C is the following:

THEOREM D. *Let $f:(B,g) \to U \subset (N,h)$ be an isometric immersion into an open set supporting a positive conformal vector field X and let $n = \dim B = \dim N \geq 3$. Assume that N is Einstein with scalar curvature $\mathcal{R}$, that $\tilde{f}: M \to N$ has constant scalar curvature $\mathcal{S}$ satisfying* (1.4) *and that there exists at least one elliptic point on M. Then the solution t of the Poisson problem* ($\mathcal{P}$) *satisfies*

$$\frac{V(M)}{V(B)} \geq \frac{n}{(n-1)^{1/2}(n-2)^{1/2}}\left(\mathcal{S} - \frac{n-2}{n}\mathcal{R}\right)^{1/2} + \frac{b}{a}\,\frac{2\mathcal{R}E(t)}{n-1}\,\frac{V(M)}{V(B)^2} \tag{1.6}$$

and $\int_M \nu_M(t)^2\, dV_{\tilde{g}} > 0$. As a consequence, if t satisfies

$$\mathcal{R}\,\frac{2E(t)}{\int_M \nu_M(t)^2\, dV_{\tilde{g}}} \geq \frac{b-\tilde{a}}{\tilde{a}}\, n\left(\frac{n-1}{n-2}\right)^{1/2}\left(\mathcal{S} - \frac{n-2}{n}\mathcal{R}\right)^{1/2} \tag{1.7}$$

then $\tilde{f}: M \to N$ is totally umbilical.

Observe that when $(N,h) = (\mathbf{R}^n, \langle\ ,\ \rangle)$ then automatically on M there must exist at least an elliptic point, so that, since $\mathcal{R} = 0$, condition (1.4) is obviously met. Choosing $U = \mathbf{R}^n$ and X to be the position vector field, (1.7) is vacuously satisfied so that we conclude that $\tilde{f}: M \to \mathbf{R}^n$ is totally umbilical, recovering the result of Ros.

We now indicate by ω_m the volume of the canonical m-sphere in $\mathbf{R}^{m+1}$ and with $C(B,M)$ the Croke isoperimetric constant [C] relative to B and its boundary M. As another consequence of Proposition C we obtain:

THEOREM E. *Let $f:(B,g) \to U \subset (N,h)$ be an isometric immersion into an open set supporting a positive conformal vector field X and let $n = \dim B = \dim N \geq 3$. Assume that N is Einstein with scalar curvature $\mathcal{R} \geq 0$, that $\tilde{f}: M \to N$ has constant scalar curvature $\mathcal{S}$ satisfying* (1.4) *and that there exists at least one elliptic point on M. Then*

$$C(B,M)^{(n+1)/(n-1)} \leq \frac{b}{\tilde{a}}\,\frac{n}{2}\,\frac{\omega_n}{\omega_{n-1}}, \tag{1.8}$$

and if equality holds M is conformally diffeomorphic to a standard sphere.

Note that if B is embedded in $\mathbf{R}^n$, then

$$C(B,M)^{(n+1)/(n-1)} = \frac{n}{2}\frac{\omega_n}{\omega_{n-1}}$$

as immediately follows from Reid's isoperimetric inequality [R]. To achieve the first conclusion of the above theorem we make essential use of the solution of the Yamabe problem due to Aubin [A] and Schoen [S], together with a result of Obata asserting the uniqueness, unless $(M,\tilde{g})$ is isometric to a standard sphere, of a metric with constant scalar curvature in a conformal class admitting an Einstein representative, on a compact manifold [O2].

2. Proof of the theorems and further results

Let $f:(B,g)\to U\subset(N,h)$ be an isometric immersion, X a vector field on U and let $X_f = X\circ f$. Indicate by $\{X_i\}$, $i=1,\ldots,m=\dim B$, a (local) orthonormal frame on B and with H_f the mean-curvature vector of the immersion. Then an elementary computation gives the following formula:

$$h(mH_f,X_f) = \operatorname{div}_g W - \sum_{i=1}^{m}\frac{1}{2}L_X h\big(df(X_i),df(X_i)\big), \tag{2.1}$$

where W is the vector field on B defined by $g(W,Y)=h(df(Y),X_f)$, for each vector field Y on B. Let M be the (possibly empty) boundary of B and $\tilde{g}$ the induced metric on it. Indicating by ν_M the outward unit normal of M with respect to B, applying property (1.1) of almost conformal vector fields and Stokes' theorem to (2.1) we obtain:

Proposition 1. *Let $f:(B,g)\to U\subset(N,h)$ be an isometric immersion into an open set supporting an almost conformal vector field X and let $m=\dim B$. Then*

$$aV(B)\le -\int_B h(X_f,H_f)\,dV_g + \frac{1}{m}\int_M h\big(X_{\tilde{f}},df(\nu_M)\big)\,dV_{\tilde{g}} \le bV(B). \tag{2.2}$$

Remarks. (1) In case X is a homothetic vector field and $M=\varnothing$, (2.2) becomes

$$aV(B) = -\int_B h(X_f,H_f)\,dV_g,$$

which, for X the position vector field in $\mathbf{R}^n$, is a classical formula of Minkowski for convex bodies.

(2) Inequalities (2.2) immediately imply that there are no minimal immersions of a compact manifold without boundary into an open set U supporting a posi-

tive almost conformal vector field. It seems worthwhile to generalize this fact to the following:

Proposition 2. *Let* (B,g) *be a complete, non-compact, oriented Riemannian manifold,* $f:(B,g)\to U\subset(N,h)$ *a minimal immersion into an open set supporting a vector field* X *satisfying* $L_X h\geq 2\alpha h$, $\inf_U \alpha>0$. *Let* p *be some fixed point in* B *and* ρ *the distance function from* p. *Then*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int_{B_\rho(p)}\|X_f\|\,dV_g\neq 0.$$

Furthermore, if there exists a real number $r>1$ *such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho^r}\int_{B_\rho(p)}\|X_f\|^r\,dV_g=0$$

then B *has infinite volume.*

The argument follows [R–S]; we report it here in this new setting for the sake of completeness.

Proof. Assume first that $V(B)<\infty$ and $r>1$. Let ω be the dual form to the vector field W appearing in (2.1), that is, $\omega(Y)=g(W,Y)$ and let $\star$ be the Hodge star operator on B. Then $d\star\omega=\operatorname{div}_g(W)\,dV_g$ and since f is minimal, (2.1) gives

$$m\alpha\,dV_g=d\star\omega, \tag{j}$$

where $m=\dim B$. Let $|\star\omega|$ be the Hilbert–Schmidt norm of the $(m-1)$-form $\star\omega$. Then $|\star\omega|=|\omega|=\|W\|$, but from the definition of W and Schwartz' inequality, $\|W\|\leq\|X_f\|$. Therefore, integrating on the geodesic ball $B_\rho(p)$ and applying Holder's inequality we have

$$\frac{1}{\rho}\int_{B_\rho(p)}|\star\omega|\,dV_g\leq\left\{\int_{B_\rho(p)}1\,dV_g\right\}^{1/r'}\left\{\frac{1}{\rho^r}\int_{B_\rho(p)}\|X_f\|^r\,dV_g\right\}^{1/r},$$

where r' is the exponent conjugate to r, and since $V(B)<\infty$,

$$\frac{1}{\rho}\int_{B_\rho(p)}|\star\omega|\,dV_g\leq V(B)^{1/r'}\left\{\frac{1}{\rho^r}\int_{B_\rho(p)}\|X_f\|^r\,dV_g\right\}^{1/r}\to 0,\qquad\text{as }\rho\to\infty.$$

By the Gaffney–Yau extension of Stokes' theorem, see the appendix of [Y], there exists a sequence of compact domains, D_i, in B such that $D_i\subset D_{i+1}$,

$\bigcup_{i=1}^{\infty} D_i = B$ and $\int_{D_i} d \star w \to 0$ as $i \to \infty$. This together with (j) and the fact that $\inf_U \alpha > 0$ gives

$$V(B) = \lim_{i \to \infty} \int_{D_i} dV_g = 0,$$

a contradiction. This proves the second part of Proposition 2. The first part is proved in a similar way, by assuming that

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_{B_\rho(p)} \|X_f\| \, dV_g = 0,$$

and using the above Holder's inequality with $r = 1$ and $1/r' = 0$ obtaining a contradiction, without needing the assumption $V(B) < \infty$. ■

As another consequence of Proposition 1 we give a proof of Proposition A.

Proof of Proposition A. Let $\tilde{\nu}_M$ denote $df(\nu_M)$. From Proposition 1 with $m = n$ we have

$$\text{(i)} \qquad 0 < aV(B) \leq \frac{1}{n} \int_M h(X_{\tilde{f}}, \tilde{\nu}_M) \, dV_{\tilde{g}} \leq bV(B).$$

Analogously, considering $\tilde{f}: M \to U \subset (N, h)$, since $\partial M = \emptyset$, we have

$$\text{(ii)} \qquad 0 < \tilde{a}V(M) \leq -\int_M h(X_{\tilde{f}}, \tilde{H}) \, dV_{\tilde{g}} \leq \tilde{b}V(M).$$

But $\tilde{H}$ is parallel to $\tilde{\nu}_M$ so that

$$\text{(iii)} \qquad h(X_{\tilde{f}}, \tilde{H}) = h(\tilde{H}, \tilde{\nu}_M) h(X_{\tilde{f}}, \tilde{\nu}_M)$$

with $h(\tilde{H}, \tilde{\nu}_M)$ a fixed real number due to the constancy of the mean curvature. Putting (ii) and (iii) together gives

$$0 < \tilde{a}V(M) \leq -h(\tilde{H}, \tilde{\nu}_M) \int_M h(X_{\tilde{f}}, \tilde{\nu}_M) \, dV_{\tilde{g}}$$

and the first part of (i) implies that $-h(\tilde{H}, \tilde{\nu}_M)$ is positive, and so $\|\tilde{H}\| = -h(\tilde{H}, \tilde{\nu}_M)$. We can thus rewrite (ii) as

$$\text{(iv)} \qquad \tilde{a}V(M) \leq \|\tilde{H}\| \int_M h(X_{\tilde{f}}, \tilde{\nu}_M) \, dV_{\tilde{g}} \leq \tilde{b}V(M).$$

Using the second half of (i) and the first half of (iv) we obtain

$$\text{(v)} \qquad \tilde{a}V(M) \leq nb\|\tilde{H}\|V(B),$$

and similarly

$$\text{(vi)} \qquad aV(B)\|\tilde{H}\| \leq \frac{1}{n}\tilde{b}V(M),$$

achieving the proof of (1.2). ∎

The proof of Theorem B is a combination of formula (2.2) with ideas of Reilly [RR1]. First we recall his local Euclidean analogue [RR2] of Obata's characterization of the sphere [O1].

LEMMA (Reilly). *Let (B,g) be a compact manifold admitting a function $v: B \to \mathbf{R}$ and a non-zero constant Λ such that*

(i) $\operatorname{Hess}(v) = \Lambda g$,

(ii) $v_{/M}$ *is constant.*

Then B is isometric to a (metric) ball in $\mathbf{R}^n$, $n = \dim B$.

A second ingredient is the following integral formula [RR1], where $t: B \to \mathbf{R}$ is a smooth function, $z = t_{/M}$, $u = \nu_M(t): M \to \mathbf{R}$ and where the Laplace-Beltrami operators and gradients have to be considered on the appropriate manifolds:

$$\int_B \{(\Delta t)^2 - |\operatorname{Hess}(t)|^2\}\, dV_g = \int_M \{u\Delta z - \tilde{g}(\nabla z, \nabla u) - h(\Pi(\nabla z, \nabla z), \nu_M))\}\, dV_{\tilde{g}}$$
$$- \int_M (n-1) g(H, \nu_M) u^2\, dV_{\tilde{g}}$$
$$+ \int_B \operatorname{Ricci}_B(\nabla t, \nabla t)\, dV_g.$$

Here Π and H are the second fundamental tensor and mean-curvature vector of M into B, while Ricci_B denotes the Ricci tensor of B. In case $f: (B,g) \to (N,h)$ is an isometric immersion, $n = \dim B = \dim N$ and t satisfies $t_{/M} = z = 0$, the above formula reduces to

$$(2.3)\qquad \int_B \{(\Delta t)^2 - |\mathrm{Hess}(t)|^2\}\,dV_g = -(n-1)\int_M h(\tilde{H},\tilde{\nu}_M)u^2\,dV_{\tilde{g}} + \int_B \mathrm{Ricci}_N(\nabla t,\nabla t)\,dV_g.$$

Let us now consider the assumptions of Theorem B.

PROOF OF THEOREM B. Formula (2.3) together with the constancy of the mean curvature of M and Newton's inequality

$$(2.4)\qquad (\Delta t)^2 \geq \frac{n}{n-1}\{(\Delta t)^2 - |\mathrm{Hess}(t)|^2\},$$

and the fact that N is Einstein, gives for the solution t of $(\mathcal{P})$:

$$\frac{a}{b}V(B) = \int_B (\Delta t)^2\,dV_g \geq \frac{n}{n-1}\int_B \{(\Delta t)^2 - |\mathrm{Hess}(t)|^2\}\,dV_g$$

$$(2.5)\qquad = -nh(\tilde{H},\tilde{\nu}_M)\int_M u^2\,dV_{\tilde{g}} + \frac{2\mathcal{R}}{n-1}E(t).$$

On the other hand, from Holder's inequality and Stokes' theorem

$$(2.6)\qquad \int_M u^2\,dV_{\tilde{g}} \geq \frac{1}{V(M)}\left\{\int_M u\,dV_{\tilde{g}}\right\}^2 = \frac{1}{V(M)}\left\{\int_B \Delta t\,dV_g\right\}^2 = \frac{a}{b}\frac{V(B)^2}{V(M)}.$$

From (v) in the proof of Proposition A we obtain

$$aV(M) \leq \tilde{a}V(M) \leq -nbh(\tilde{H},\tilde{\nu}_M)V(B)$$

so that

$$(2.7)\qquad V(M) \leq -n\frac{b}{a}h(\tilde{H},\tilde{\nu}_M)V(B).$$

Therefore combining with (2.6) we have

$$-nh(\tilde{H},\tilde{\nu}_M)\int_M u^2\,dV_{\tilde{g}} \geq \frac{a^2}{b^2}V(B).$$

This latter together with (2.5) gives

$$V(B) \geq \frac{a}{b}V(B) + \frac{b}{a}\frac{2\mathcal{R}}{n-1}E(t),$$

that is (1.3). If equality holds in (1.3), then we have equality in (2.5) and consequently equality holds in (2.4). But this latter case happens if and only if Hess(t) is a non-zero (since t is a solution of ($\mathcal{P}$)) constant multiple of the metric. This together with Reilly's lemma completes the proof of the theorem. ■

In what follows we will make use of the following: Let $\tilde{f}: (M, \tilde{g}) \to U \subset (N, h)$ be a hypersurface isometrically immersed in the open set U and let $\{e_i\}$, $i = 1, \ldots, n-1$ be a (local) orthonormal frame of TM and e_n a unit normal to $d\tilde{f}(TM)$ globally defined on M. Indicating by $\tilde{\Pi}$ the second fundamental tensor of $\tilde{f}$ we let

$$h_{ij} = h\big(\tilde{\Pi}(e_i, e_j), e_n\big), \qquad i, j = 1, \ldots, n-1$$

be its coefficients in the given frame. Set $\tilde{H}$ for the mean-curvature vector of $\tilde{f}$ and, to simplify notation, for a given vector field X on U set $X_{\tilde{f}}^T = \sum_{i=1}^{n-1} h(X_{\tilde{f}}, d\tilde{f}(e_i))\, d\tilde{f}(e_i)$ for the projection of $X_{\tilde{f}} \in C^\infty(\tilde{f}^{-1}TN)$ onto $d\tilde{f}(TM)$. With the above notation we have

Lemma 1. *Let X be any vector field defined on U and let Y be the vector field on M given by*

$$Y = \sum_{i,j=1}^{n-1} \big\{ h_{ij} h\big(X_{\tilde{f}}, d\tilde{f}(e_j)\big) - h_{jj} h\big(X_{\tilde{f}}, d\tilde{f}(e_i)\big) \big\} e_i. \tag{2.8}$$

Then Y is globally defined and

$$\begin{aligned} \operatorname{div}_{\tilde{g}}(Y) = {} & h(X_{\tilde{f}}, e_n)\{\|\tilde{\Pi}\|^2 - (n-1)^2 \|\tilde{H}\|^2\} - \operatorname{Ricci}_N(X_{\tilde{f}}^T, e_n) \\ & + \frac{1}{2} \sum_{i,j=1}^{n-1} h_{ij} L_X h\big(d\tilde{f}(e_i), d\tilde{f}(e_j)\big) \\ & - \frac{n-1}{2} h\big(\tilde{H}, e_n\big) \sum_{i=1}^{n-1} L_X h\big(d\tilde{f}(e_i), d\tilde{f}(e_i)\big). \end{aligned} \tag{2.9}$$

The proof is just a simple but tedious calculation, which we therefore omit. In the same notation, letting $\mathcal{S}$ and $\mathcal{R}$ be the scalar curvatures of M and N respectively, we have from Gauss' equation

$$\mathcal{S} = \mathcal{R} - 2\,\operatorname{Ricci}_N(e_n, e_n) + (n-1)^2 \|\tilde{H}\|^2 - \|\tilde{\Pi}\|^2. \tag{2.10}$$

The following estimates will be crucial.

Lemma 2. *Let $f: (B, g) \to U \subset (N, h)$ be an isometric immersion with M the boundary of B, $f_{/M} = \tilde{f}$, $n = \dim B = \dim N$ and let $e_n = \tilde{\nu}_M = df(\nu_M)$ as defined above.*

(i) *If X is a conformal vector field then*

$$(2.11)\quad \begin{aligned} -(n-1)(n-2)\int_M \alpha h(\tilde{H},\tilde{\nu}_M)\,dV_{\tilde{g}} - \int_M \mathrm{Ricci}_N(X_{\tilde{f}}^T,\tilde{\nu}_M)\,dV_{\tilde{g}} \\ = \int_M h(X_{\tilde{f}},\tilde{\nu}_M)\{\mathcal{S} - \mathcal{R} + 2\,\mathrm{Ricci}_N(\tilde{\nu}_M,\tilde{\nu}_M)\}\,dV_{\tilde{g}}. \end{aligned}$$

(ii) *If X is a positive almost conformal vector field and if M is a convex hypersurface of B, i.e. for each $Z \in C^\infty(TM)$, $-h(\tilde{\Pi}(Z,Z),\tilde{\nu}_M) \geq 0$, then*

$$(2.12)\quad \begin{aligned} -(n-1)\int_M \{(n-1)\alpha - \beta\}h(\tilde{H},\tilde{\nu}_M)\,dV_{\tilde{g}} - \int_M \mathrm{Ricci}_N(X_{\tilde{f}}^T,\tilde{\nu}_M)\,dV_{\tilde{g}} \\ \leq \int_M h(X_{\tilde{f}},\tilde{\nu}_M)\{\mathcal{S} - \mathcal{R} + 2\,\mathrm{Ricci}_N(\tilde{\nu}_M,\tilde{\nu}_M)\}\,dV_{\tilde{g}}. \end{aligned}$$

PROOF. (i) We consider formula (2.9) and, since X is conformal, we estimate

$$\frac{1}{2}\sum_{i,j=1}^{n-1} h_{ij}L_X h\big(d\tilde{f}(e_i),d\tilde{f}(e_j)\big) - \frac{n-1}{2}h(\tilde{H},\tilde{\nu}_M)\sum_{i=1}^{n-1} L_X h\big(d\tilde{f}(e_i),d\tilde{f}(e_i)\big)$$
$$= -(n-1)(n-2)\alpha h(\tilde{H},\tilde{\nu}_M).$$

We substitute into (2.9), use (2.10) and integrate over M to obtain (2.11).

(ii) We estimate the same term as above from below by diagonalizing $-h(\tilde{\Pi},\tilde{\nu}_M)$ and using the fact that X is positive almost conformal. With this process via (2.9) we obtain

$$\begin{aligned} \mathrm{div}_{\tilde{g}}(Y) \geq\ & h(X_{\tilde{f}},e_n)\{\|\tilde{\Pi}\|^2 - (n-1)^2\|\tilde{H}\|^2\} \\ & - \mathrm{Ricci}_N(X_{\tilde{f}}^T,\tilde{\nu}_M) - (n-1)\{(n-1)\alpha - \beta\}h(\tilde{H},\tilde{\nu}_M), \end{aligned}$$

and (2.12) follows by integration and (2.10). ■

PROPOSITION C′. *Let $f\colon (B,g) \to U \subset (N,h)$ be an isometric immersion of the manifold B with convex boundary M (in the sense of Lemma 2 (ii)) into an Einstein manifold with scalar curvature $\mathcal{R}$. Let X be a positive almost conformal vector field on U such that $(n-1)\tilde{a} - \tilde{b} > 0$ where $n = \dim B = \dim N \geq 3$ and $\tilde{a},\tilde{b},b$ as defined in the Introduction. Assume that the scalar curvature $\mathcal{S}$ of M is a constant satisfying* (1.4). *Then the following isoperimetric inequality holds:*

$$(2.13)\qquad \frac{V(M)}{V(B)} \leq \left(\frac{n-2}{n-1}\right)^{1/2}\frac{nb}{(n-1)\tilde{a}-\tilde{b}}\left(\mathcal{S} - \frac{n-2}{n}\mathcal{R}\right)^{1/2}.$$

If equality holds then $\tilde{f}\colon M \to N$ is totally umbilical.

PROOF. In the assumptions of the proposition we use (2.12) to obtain

$$\text{(j)}\qquad (n-1)\{(n-1)\tilde{a}-\tilde{b}\}\int_M -h(\tilde{H},\tilde{\nu}_M)\,dV_{\tilde{g}} \le \int_M h(X_{\tilde{f}},\tilde{\nu}_M)\left(\mathcal{S}-\frac{n-2}{n}\,\mathcal{R}\right)dV_{\tilde{g}}.$$

On the other hand, (2.2) of Proposition 1 gives

$$\int_M h(X_{\tilde{f}},\tilde{\nu}_M)\,dV_{\tilde{g}} \le nbV(B),$$

so that together with (1.4) and (j) we obtain

$$\text{(jj)}\qquad \left(\frac{n-1}{n-2}\right)^{1/2}\{(n-1)\tilde{a}-\tilde{b}\}\int_M -(n-1)^{1/2}(n-2)^{1/2}h(\tilde{H},\tilde{\nu}_M)\,dV_{\tilde{g}} \le \left(\mathcal{S}-\frac{n-2}{n}\,\mathcal{R}\right)nbV(B).$$

Now by (2.10) and Newton's inequalities we get

$$\text{(2.14)}\qquad \left(\mathcal{S}-\frac{n-2}{n}\,\mathcal{R}\right)^{1/2} \le (n-1)^{1/2}(n-2)^{1/2}\{-h(\tilde{H},\tilde{\nu}_M)\}$$

and (jj) implies

$$\text{(jjj)}\qquad \left(\frac{n-1}{n-2}\right)^{1/2}\{(n-1)\tilde{a}-\tilde{b}\}V(M) \le nb\left(\mathcal{S}-\frac{n-2}{n}\,\mathcal{R}\right)^{1/2}V(B),$$

from which (2.13) follows immediately. Observe that equality in (2.13) implies equality in (2.14) so that in this case $\tilde{f}: M \to N$ is totally umbilical. ■

The proof of Proposition C is completely similar using (2.11) instead of (2.12). Anyway we preliminarily observe that Newton's inequalities and (1.4) imply $h(\tilde{H},\tilde{\nu}_M) \neq 0$ on M. The existence of at least an elliptic point and connectedness of M therefore imply $h(\tilde{H},\tilde{\nu}) < 0$ on M, so that instead of (jj) we obtain

$$(n-1)^{1/2}(n-2)^{1/2}\tilde{a}\int_M -h(\tilde{H},\tilde{\nu}_M)\,dV_{\tilde{g}} \le \left(\mathcal{S}-\frac{n-2}{n}\,\mathcal{R}\right)nbV(B),$$

from which equation (1.5) follows.

PROOF OF THEOREM D. As in the proof of Theorem B we obtain

$$\text{(k)}\qquad \int_B (\Delta t)^2\, dV_g \geq n \int_M -h(\tilde{H}, \tilde{\nu}_M) u^2\, dV_{\tilde{g}} + \frac{2\mathcal{R}}{n-1} E(t),$$

for $t: B \to \mathcal{R}$ satisfying $t_{/M} \equiv 0$, $u = \nu_M(t)$. As in Proposition C, $h(\tilde{H}, \tilde{\nu}_M) < 0$, so that (2.14) holds. From Holder's inequality

$$\int_M u\, dV_{\tilde{g}} \leq \left\{ \int_M u^2\, dV_{\tilde{g}} \right\}^{1/2} V(M)^{1/2}$$

and (2.14) we obtain

$$\begin{aligned} \int_M -h(\tilde{H}, \tilde{\nu}_M) u^2\, dV_{\tilde{g}} &\geq \frac{1}{(n-1)^{1/2}(n-2)^{1/2}} \left(\mathcal{S} - \frac{n-2}{n} \mathcal{R} \right)^{1/2} \int_M u^2\, dV_{\tilde{g}} \\ &\geq \frac{1}{(n-1)^{1/2}(n-2)^{1/2} V(M)} \left(\mathcal{S} - \frac{n-2}{n} \mathcal{R} \right)^{1/2} \\ &\quad \times \left\{ \int_M u\, dV_{\tilde{g}} \right\}^2, \end{aligned}$$

and applying Stokes' theorem

$$\begin{aligned} \text{(kk)}\quad \int_M -h(\tilde{H}, \tilde{\nu}_M) u^2\, dV_{\tilde{g}} &\geq \frac{1}{(n-1)^{1/2}(n-2)^{1/2} V(M)} \left(\mathcal{S} - \frac{n-2}{n} \mathcal{R} \right)^{1/2} \\ &\quad \times \left\{ \int_B \Delta t\, dV_g \right\}^2. \end{aligned}$$

Putting together (k) and (kk) we have

$$\begin{aligned} \int_B (\Delta t)^2\, dV_g \geq{}& \frac{n}{(n-1)^{1/2}(n-2)^{1/2}} \left(\mathcal{S} - \frac{n-2}{n} \mathcal{R} \right)^{1/2} \frac{1}{V(M)} \left\{ \int_B \Delta t\, dV_g \right\}^2 \\ &+ \frac{2\mathcal{R}}{n-1} E(t). \end{aligned}$$

Assuming t to be the solution of $(\mathcal{P})$ we therefore obtain

$$\frac{a}{b} V(B) \geq \frac{n}{(n-1)^{1/2}(n-2)^{1/2}} \left(\mathcal{S} - \frac{n-2}{n} \mathcal{R} \right)^{1/2} \frac{a}{b} \frac{V(B)^2}{V(M)} + \frac{2\mathcal{R}}{n-1} E(t),$$

and (kk) together with (1.4) give $\int_M u^2 \, dV_{\tilde{g}} > 0$, which could be equally obtained from (2.6), which holds in the present assumptions. Now (1.6) follows immediately. Using now (2.6) we obtain from (1.6).

$$\frac{V(M)}{V(B)} \geq \frac{n}{(n-1)^{1/2}(n-2)^{1/2}}\left(\mathcal{S} - \frac{n-2}{n}\,\mathcal{R}\right)^{1/2} + \frac{2\mathcal{R}E(t)}{n-1}\,\frac{1}{\int_M u^2 dV_{\tilde{g}}},$$

and if (1.7) holds this latter implies equality in (1.5) showing that $\tilde{f}: M \to N$ has to be totally umbilical. ∎

Observe that the first part of the theorem is independent of the existence of X and holds for t the solution of ($\mathcal{P}$) with a/b any positive constant. The existence and properties of X are used in the second part via Proposition C to show that (1.7) is a sufficient condition to guarantee that $\tilde{f}: M \to N$ is totally umbilical. In this respect we could have used Proposition C′ having appropriately modified (1.7). An analogous observation applies to Theorem E that we are now going to prove.

PROOF OF THEOREM E. We first recall that Croke in [C] established the following isoperimetric inequality:

$$\text{(a)} \qquad \frac{V(M)^n}{V(B)^{n-1}} \geq 2^{n-1} C^{n+1} \frac{\omega_{n-1}^n}{\omega_n^{n-1}},$$

where from now on, to simplify notation, C indicates $C(B,M)$, the Croke constant of the introduction. Setting

$$V(B) = \frac{1}{n}\,\omega_{n-1} A$$

for some positive constant A, (a) gives

$$\text{(aa)} \qquad V(M) \geq \left(\frac{2A}{n}\right)^{(n-1)/n} C^{(n+1)/n} \left\{\frac{\omega_{n-1}}{\omega_n}\right\}^{(n-1)/n} \omega_{n-1}.$$

Since $\mathcal{R} \geq 0$ from (1.5) of Proposition C we have

$$\frac{\tilde{a}^2}{b^2}\,\frac{(n-1)(n-2)}{n^2}\,\frac{V(M)^2}{V(B)^2} \leq \mathcal{S},$$

and therefore

$$\mathcal{S} \geq (n-1)(n-2)\left(\frac{\tilde{a}}{b}\right)^2 \left\{ \frac{V(M)}{A\omega_{n-1}} \right\}^2 .$$

Using (aa) we thus obtain

$$\mathcal{S} \geq \frac{1}{A^2}\left(\frac{2A}{n}\right)^{2(n-1)/n}\left(\frac{\tilde{a}}{b}\right)^2 (n-1)(n-2)C^{2(n+1)/n}\left\{\frac{\omega_{n-1}}{\omega_n}\right\}^{2(n-1)/n} .$$

Multiplying both terms by $V(M)^{2/(n-1)}$ and applying again (aa) we get

$$\text{(b)} \qquad \frac{4}{n^2} C^{2(n+1)/(n-1)} \omega_{n-1}^{2/(n-1)} \left\{\frac{\omega_{n-1}}{\omega_n}\right\}^2 (n-1)(n-2)\left(\frac{\tilde{a}}{b}\right)^2 \leq \mathcal{S}V(M)^{2(n-1)} .$$

Observe now that since $(M,\tilde{g})$ is Einstein, by a result of Obata [O2], either $(M,\tilde{g})$ is isometric to the standard sphere $\mathbb{S}^{n-1}$ or $\tilde{g}$ is the unique metric with constant scalar curvature in the conformal class $[\tilde{g}]$ determined by $\tilde{g}$ itself. This implies by results of Aubin [A] and Schoen [S] that $\mathcal{S}V(M)^{2/(n-1)}$ is the Yamabe constant associated to $(M,[\tilde{g}])$. As such, it always satisfies

$$\text{(bb)} \qquad \mathcal{S}V(M)^{2/(n-1)} \leq (n-1)(n-2)\omega_{n-1}^{2/(n-1)},$$

with equality holding if and only if $(M,\tilde{g})$ is conformally diffeomorphic to the standard sphere $\mathbb{S}^{n-1}$ [S]. Now, (b) and (bb) imply (1.8) and complete the proof of the theorem. ■

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